

QUASI-MINIMAL LORENTZ SURFACES WITH POINTWISE 1-TYPE GAUSS MAP IN PSEUDO-EUCLIDEAN 4-SPACE

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ABSTRACT. A Lorentz surface in the four-dimensional pseudo-Euclidean space with neutral metric is called quasi-minimal if its mean curvature vector is lightlike at each point. In the present paper we obtain the complete classification of quasi-minimal Lorentz surfaces with pointwise 1-type Gauss map.

1. INTRODUCTION

In the present paper we study Lorentz surfaces in pseudo-Euclidean space \mathbb{E}_2^4 . A surface is called *minimal* if its mean curvature vector vanishes identically. Minimal surfaces are important in differential geometry as well as in physics. Minimal Lorentz surfaces in \mathbb{C}_1^2 have been classified recently by B.-Y. Chen [10]. Several classification results for minimal Lorentz surfaces in indefinite space forms are obtained in [16]. In particular, a complete classification of all minimal Lorentz surfaces in a pseudo-Euclidean space \mathbb{E}_s^m with arbitrary dimension m and arbitrary index s is given.

A natural extension of minimal surfaces are quasi-minimal surfaces. A surface in a pseudo-Riemannian manifold is called *quasi-minimal* (also pseudo-minimal or marginally trapped) if its mean curvature vector is lightlike at each point of the surface [33]. Quasi-minimal surfaces in pseudo-Euclidean space have been very actively studied in the last few years. In [7] B.-Y. Chen classified quasi-minimal Lorentz flat surfaces in \mathbb{E}_2^4 and gave a complete classification of biharmonic Lorentz surfaces in \mathbb{E}_2^4 with lightlike mean curvature vector. Several other families of quasi-minimal surfaces have also been classified. For example, quasi-minimal surfaces with constant Gauss curvature in \mathbb{E}_2^4 were classified in [8, 23]. Quasi-minimal Lagrangian surfaces and quasi-minimal slant surfaces in complex space forms were classified, respectively, in [19] and [21]. The classification of quasi-minimal surfaces with parallel mean curvature vector in \mathbb{E}_2^4 is obtained in [20]. In [26] the classification of quasi-minimal rotational surfaces of elliptic, hyperbolic or parabolic type is given. For an up-to-date survey on quasi-minimal surfaces, see also [9].

Another basic class of surfaces in Riemannian and pseudo-Riemannian geometry are the surfaces with parallel mean curvature vector field, since they are critical points of some natural functionals and play important role in differential geometry, the theory of harmonic maps, as well as in physics. Surfaces with parallel mean curvature vector field in Riemannian space forms were classified in the early 1970s by Chen [3] and Yau [36]. Recently, spacelike surfaces with parallel mean curvature vector field in arbitrary indefinite space forms were classified in [11] and [12]. A complete classification of Lorentz surfaces with parallel mean curvature vector field in arbitrary pseudo-Euclidean space \mathbb{E}_s^m is given in [13, 25, 27]. A survey on classical and recent results concerning submanifolds with parallel mean curvature vector in Riemannian manifolds as well as in pseudo-Riemannian manifolds is presented in [14].

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The study of submanifolds of Euclidean or pseudo-Euclidean space via the notion of finite type immersions began in the late 1970's with the papers [4, 5] of B.-Y. Chen. An isometric immersion $x : M \rightarrow \mathbb{E}^m$ of a submanifold M in Euclidean m -space \mathbb{E}^m (or pseudo-Euclidean space \mathbb{E}_s^m) is said to be of *finite type* [4], if x identified with the position vector field of M in \mathbb{E}^m (or \mathbb{E}_s^m) can be expressed as a finite sum of eigenvectors of the Laplacian Δ of M , i.e.

$$x = x_0 + \sum_{i=1}^k x_i,$$

where x_0 is a constant map, x_1, x_2, \dots, x_k are non-constant maps such that $\Delta x_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, $1 \leq i \leq k$. More precisely, if $\lambda_1, \lambda_2, \dots, \lambda_k$ are different, then M is said to be of *k-type*. Many results on finite type immersions have been collected in the survey paper [6]. The newest results on submanifolds of finite type are collected in [17].

The notion of finite type immersion is naturally extended to the Gauss map G on M by B.-Y. Chen and P. Piccinni in [22], where they introduced the problem “*To what extent does the type of the Gauss map of a submanifold of \mathbb{E}^m determine the submanifold?*”. A submanifold M of an Euclidean (or pseudo-Euclidean) space is said to have *1-type Gauss map* G , if G satisfies $\Delta G = a(G + C)$ for some $a \in \mathbb{R}$ and some constant vector C .

A submanifold M is said to have *pointwise 1-type Gauss map* if its Gauss map G satisfies

$$(1) \quad \Delta G = \phi(G + C)$$

for some non-zero smooth function ϕ on M and some constant vector C [18]. A pointwise 1-type Gauss map is called *proper* if the function ϕ is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of *first kind* if the vector C is zero. Otherwise, it is said to be of *second kind*.

Classification results on surfaces with pointwise 1-type Gauss map in Minkowski space have been obtained in the last few years. For example, in [30] Y. Kim and D. Yoon studied ruled surfaces with 1-type Gauss map in Minkowski space \mathbb{E}_1^m and gave a complete classification of null scrolls with 1-type Gauss map. The classification of ruled surfaces with pointwise 1-type Gauss map of first kind in Minkowski space \mathbb{E}_1^3 is given in [28]. Ruled surfaces with pointwise 1-type Gauss map of second kind in Minkowski 3-space were classified in [24].

The complete classification of flat rotation surfaces with pointwise 1-type Gauss map in the 4-dimensional pseudo-Euclidean space \mathbb{E}_2^4 is given in [29]. A classification of flat Moore type rotational surfaces in terms of the type of their Gauss map is obtained in [1]. Recently, Arslan and the first author have obtained a classification of meridian surfaces with pointwise 1-type Gauss map [2]. The classification of marginally trapped surfaces with pointwise 1-type Gauss map in Minkowski 4-space is given in [31] and [34].

In the present paper we study quasi-minimal Lorentz surfaces in \mathbb{E}_2^4 with pointwise 1-type Gauss map. First we describe the quasi-minimal surfaces with harmonic Gauss map proving that each such surface is a flat surface with parallel mean curvature vector field. Next we give explicitly all flat quasi-minimal surfaces with pointwise 1-type Gauss map (Theorem 3.7). Further, we obtain that a non-flat quasi-minimal surface with flat normal connection has pointwise 1-type Gauss map if and only if it has parallel mean curvature vector field (Theorem 3.8). We give necessary and sufficient conditions for a quasi-minimal surface with non-flat normal connection to have pointwise 1-type Gauss map. In Theorem 3.12 we present the complete classification of quasi-minimal surface with non-flat normal connection and pointwise 1-type Gauss map. At the end of the paper we give an explicit example of a quasi-minimal surface with non-flat normal connection and pointwise 1-type

Gauss map. This is also an example of a quasi-minimal surface with non-parallel mean curvature vector field.

2. PRELIMINARIES

Let \mathbb{E}_s^m be the pseudo-Euclidean m -space endowed with the canonical pseudo-Euclidean metric of index s given by

$$g_0 = \sum_{i=1}^{m-s} dx_i^2 - \sum_{j=m-s+1}^m dx_j^2,$$

where x_1, x_2, \dots, x_m are rectangular coordinates of the points of \mathbb{E}_s^m . As usual, we denote by $\langle \cdot, \cdot \rangle$ the indefinite inner scalar product with respect to g_0 .

A non-zero vector v is said to be *spacelike* (respectively, *timelike*) if $\langle v, v \rangle > 0$ (respectively, $\langle v, v \rangle < 0$). A vector v is called *lightlike* if it is nonzero and satisfies $\langle v, v \rangle = 0$.

We use the following denotations:

$$\mathbb{S}_s^{m-1}(1) = \{v \in \mathbb{E}_s^m : \langle v, v \rangle = 1\},$$

$$\mathbb{H}_{s-1}^{m-1}(-1) = \{v \in \mathbb{E}_s^m : \langle v, v \rangle = -1\}.$$

$\mathbb{S}_s^{m-1}(1)$ and $\mathbb{H}_{s-1}^{m-1}(-1)$ ($m \geq 3$) are complete pseudo-Riemannian manifolds with constant sectional curvatures 1 and -1 , respectively. The pseudo-Euclidean space \mathbb{E}_1^m is known as the *Minkowski m -space*, the space $\mathbb{S}_1^{m-1}(1)$ is known as the *de Sitter space*, and the space $\mathbb{H}_1^{m-1}(-1)$ is the *hyperbolic space* (or the *anti-de Sitter space*) [32].

The Gauss map G of a submanifold M^n of \mathbb{E}_s^m is defined as follows. Let $G(n, m)$ be the Grassmannian manifold consisting of all oriented n -planes through the origin of \mathbb{E}_s^m and $\wedge^n \mathbb{E}_s^m$ be the vector space obtained by the exterior product of n vectors in \mathbb{E}_s^m . Let $e_{i_1} \wedge \dots \wedge e_{i_n}$ and $f_{j_1} \wedge \dots \wedge f_{j_n}$ be two vectors of $\wedge^n \mathbb{E}_s^m$. The indefinite inner product on the Grassmannian manifold is defined by

$$\langle e_{i_1} \wedge \dots \wedge e_{i_n}, f_{j_1} \wedge \dots \wedge f_{j_n} \rangle = \det(\langle e_{i_k}, f_{j_l} \rangle).$$

Thus, in a natural way, we can identify $\wedge^n \mathbb{E}_s^m$ with some pseudo-Euclidean space \mathbb{E}_k^N , where $N = \binom{m}{n}$, and k is a positive integer. Let $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$ be a local orthonormal frame field in \mathbb{E}_s^m such that e_1, e_2, \dots, e_n are tangent to M^n and $e_{n+1}, e_{n+2}, \dots, e_m$ are normal to M^n . The map $G : M^n \rightarrow G(n, m)$ defined by $G(p) = (e_1 \wedge e_2 \wedge \dots \wedge e_n)(p)$ is called the *Gauss map* of M^n . It is a smooth map which carries a point p of M^n into the oriented n -plane in \mathbb{E}_s^m obtained by the parallel translation of the tangent space of M^n at p in \mathbb{E}^m [29]. See also [34] for detailed information about definition and geometrical interpretation of the Gauss map of submanifolds.

For any real valued function φ on M^n the Laplacian $\Delta\varphi$ of φ is given by the formula

$$\Delta\varphi = - \sum_i \varepsilon_i (\tilde{\nabla}_{e_i} \tilde{\nabla}_{e_i} \varphi - \tilde{\nabla}_{\tilde{\nabla}_{e_i} e_i} \varphi),$$

where $\varepsilon_i = \langle e_i \wedge e_i \rangle = \pm 1$, $\tilde{\nabla}$ is the Levi-Civita connection of \mathbb{E}_s^m and ∇ is the induced connection on M^n .

In the present paper we consider the pseudo-Euclidean 4-dimensional space \mathbb{E}_2^4 with the canonical pseudo-Euclidean metric of index 2. In this case, the metric g_0 becomes $g_0 = dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$. A surface M_1^2 in \mathbb{E}_2^4 is called *Lorentz* if the induced metric g on M_1^2 is Lorentzian. So, at each point $p \in M_1^2$ we have the following decomposition

$$\mathbb{E}_2^4 = T_p M_1^2 \oplus N_p M_1^2$$

with the property that the restriction of the metric onto the tangent space $T_p M_1^2$ is of signature $(1, 1)$, and the restriction of the metric onto the normal space $N_p M_1^2$ is of signature $(1, 1)$.

We denote by ∇ and $\tilde{\nabla}$ the Levi Civita connections of M_1^2 and \mathbb{E}_2^4 , respectively. For vector fields X, Y tangent to M_1^2 and a vector field ξ normal to M_1^2 , the formulas of Gauss and Weingarten, giving a decomposition of the vector fields $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X \xi$ into tangent and normal components, are given respectively by [3]:

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y); \\ \tilde{\nabla}_X \xi &= -A_\xi X + D_X \xi.\end{aligned}$$

These formulas define the second fundamental form h , the normal connection D , and the shape operator A_ξ with respect to ξ . For each normal vector field ξ , the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_p M_1^2$ at $p \in M_1^2$. In general, A_ξ is not diagonalizable. It is well known that the shape operator and the second fundamental form are related by the formula

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle$$

for X, Y tangent to M_1^2 and ξ normal to M_1^2 .

The mean curvature vector field H of M_1^2 in \mathbb{E}_2^4 is defined as $H = \frac{1}{2} \text{tr } h$. The surface M_1^2 is called *minimal* if its mean curvature vector vanishes identically, i.e. $H = 0$. The surface M_1^2 is called *quasi-minimal* if its mean curvature vector is lightlike at each point, i.e. $H \neq 0$ and $\langle H, H \rangle = 0$. Obviously, quasi-minimal surfaces are always non-minimal.

A normal vector field ξ on M_1^2 is called *parallel in the normal bundle* (or simply *parallel*) if $D\xi = 0$ holds identically [15]. The surface M_1^2 is said to have *parallel mean curvature vector field* if its mean curvature vector H satisfies $DH = 0$ identically.

3. CLASSIFICATION OF QUASI-MINIMAL SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

In this section we study quasi-minimal surfaces in pseudo-Euclidean space \mathbb{E}_2^4 . We obtain complete classification of quasi-minimal surfaces with pointwise 1-type Gauss map.

3.1. Moving frame on a quasi-minimal surface. Let M_1^2 be a Lorentz surface in \mathbb{E}_2^4 . Then, locally there exists a coordinate system (u, v) on M_1^2 such that the metric tensor is given by

$$g = -f^2(u, v)(du \otimes dv + dv \otimes du)$$

for some positive function $f(u, v)$ [15]. Thus, putting $x = f^{-1} \frac{\partial}{\partial u}$ and $y = f^{-1} \frac{\partial}{\partial v}$, we obtain a pseudo-orthonormal frame field $\{x, y\}$ of the tangent bundle of M_1^2 such that $\langle x, x \rangle = 0$, $\langle y, y \rangle = 0$, $\langle x, y \rangle = -1$. Then the mean curvature vector field H is given by

$$H = -h(x, y).$$

Now, let M_1^2 be quasi-minimal, i.e. its mean curvature vector is lightlike at each point. Then there exists a pseudo-orthonormal frame field $\{n_1, n_2\}$ of the normal bundle such that $n_1 = -H$, $\langle n_1, n_1 \rangle = 0$, $\langle n_2, n_2 \rangle = 0$, $\langle n_1, n_2 \rangle = -1$.

By a direct computation we obtain the following derivative formulas:

$$\begin{aligned}
(2a) \quad & \tilde{\nabla}_x x = \gamma_1 x + a n_1 + b n_2, & \tilde{\nabla}_y x &= -\gamma_2 x + n_1, \\
(2b) \quad & \tilde{\nabla}_x y = -\gamma_1 y + n_1, & \tilde{\nabla}_y y &= \gamma_2 y + c n_1 + d n_2, \\
(2c) \quad & \tilde{\nabla}_x n_1 = -b y + \beta_1 n_1, & \tilde{\nabla}_y n_1 &= -d x + \beta_2 n_1, \\
(2d) \quad & \tilde{\nabla}_x n_2 = -x - a y - \beta_1 n_2, & \tilde{\nabla}_y n_2 &= -c x - y - \beta_2 n_2
\end{aligned}$$

for some smooth functions a, b, c, d, β_1 , and β_2 , where $\gamma_1 = \frac{f_u}{f^2}$ and $\gamma_2 = \frac{f_v}{f^2}$. Thus, the Gaussian curvature K and the normal curvature \varkappa of M_1^2 are

$$\begin{aligned}
(3) \quad & K = -R(x, y, y, x) = x(\gamma_2) + y(\gamma_1) + 2\gamma_1\gamma_2, \\
& \varkappa = -R^D(x, y, n_1, n_2) = x(\beta_2) - y(\beta_1) + \gamma_1\beta_2 - \gamma_2\beta_1,
\end{aligned}$$

where R and R^D are the curvature tensors associated with the connections ∇ and D , respectively.

Lemma 3.1. *If M_1^2 is a quasi-minimal surface with parallel mean curvature vector field, then M_1^2 has flat normal connection.*

Proof. It follows from (2c) that $D_x H = \beta_1 H$, $D_y H = \beta_2 H$. Hence, the mean curvature vector field H is parallel if and only if $\beta_1 = \beta_2 = 0$. Now, under the assumption $\beta_1 = \beta_2 = 0$, equality (3) implies $\varkappa = 0$. Therefore, if H is parallel then the surface has flat normal connection. \square

Using the equations of Gauss and Codazzi, from formulas (2) we obtain the following integrability conditions:

$$\begin{aligned}
(4a) \quad & x(c) = -c\beta_1 - 2c\gamma_1 + \beta_2, \\
(4b) \quad & x(d) = d\beta_1 - 2d\gamma_1, \\
(4c) \quad & y(a) = -a\beta_2 - 2a\gamma_2 + \beta_1, \\
(4d) \quad & y(b) = b\beta_2 - 2b\gamma_2, \\
(4e) \quad & x(\gamma_2) + y(\gamma_1) + 2\gamma_1\gamma_2 = ad + bc, \\
(4f) \quad & x(\beta_2) - y(\beta_1) - \beta_1\gamma_2 + \beta_2\gamma_1 = ad - bc.
\end{aligned}$$

Equalities (4e) and (4f) imply that the Gauss curvature K and the normal curvature \varkappa are expressed as follows:

$$\begin{aligned}
(5) \quad & K = ad + bc, \\
(6) \quad & \varkappa = ad - bc.
\end{aligned}$$

Remark 3.2. If $b = d = 0$, then (5) implies $K = 0$, i.e. M_1^2 is flat. In [7, Theorem 4.1], B.-Y. Chen obtained complete classification of flat quasi-minimal surfaces in \mathbb{E}_2^4 . Considering the proof of this theorem, one can see that a quasi-minimal surface in \mathbb{E}_2^4 satisfying $b = d = 0$ is congruent to a surface parametrized by

$$(7) \quad z(u, v) = \left(\theta(u, v), \frac{u-v}{\sqrt{2}}, \frac{u+v}{\sqrt{2}}, \theta(u, v) \right)$$

for a smooth function θ .

Lemma 3.3. *Let M_1^2 be a quasi-minimal surface in \mathbb{E}_2^4 with parallel mean curvature vector field. If M_1^2 has constant Gauss curvature, then M_1^2 is flat.*

Proof. Let M_1^2 be a quasi-minimal surface with parallel mean curvature vector field and constant Gaussian curvature K_0 . Assume that $K_0 \neq 0$. Since M_1^2 has parallel mean curvature vector, according to Lemma 3.1 it has flat normal connection. Therefore, because of (6), we have $ad = bc$ and hence the Gauss curvature is $K_0 = 2ad$. Since $K_0 \neq 0$ we have that a, b, c, d do not vanish and satisfy

$$(8) \quad a = \alpha c, \quad b = \alpha d$$

for a smooth function $\alpha(u, v)$. Note that α does not vanish and $\alpha cd = \text{const}$.

In addition, since M_1^2 has parallel mean curvature vector field, we have $\beta_1 = \beta_2 = 0$. Therefore, equalities (4c), (4a), (4b) take the form

$$(9a) \quad x(c) = -2c\gamma_1,$$

$$(9b) \quad x(d) = -2d\gamma_1,$$

$$(9c) \quad y(a) = -2a\gamma_2.$$

Equalities (9a), (9b) together with (8) imply that

$$(10) \quad x(\alpha) = 4\alpha\gamma_1.$$

Applying x to the first equality of (8) and using (9a) and (10) we obtain $x(a) = 2a\gamma_1$.

Having in mind that $\gamma_1 = \frac{f_u}{f^2}$, we get

$$\frac{a_u}{a} = 2\frac{f_u}{f},$$

which implies $a(u, v) = \varphi(v)f^2(u, v)$ for a smooth function $\varphi(v)$. Now using (9c) we get $\frac{\varphi'(v)}{4\varphi(v)} = -\frac{f_v}{f}$. Since φ is a function of v , we obtain $\frac{\partial}{\partial u} \left(\frac{f_v}{f} \right) = 0$. Solving this equation, we get $f(u, v) = f_1(u)f_2(v)$ for some smooth functions $f_1(u)$, $f_2(v)$. On the other hand, the Gauss curvature is expressed by the function $f(u, v)$ according to the formula $K = \frac{2ff_{uv} - 2f_u f_v}{f^4}$. Hence, $K_0 = 0$, which contradicts the assumption $K_0 \neq 0$. □

3.2. Gauss map of quasi-minimal surfaces. Let M_1^2 be a quasi-minimal surface in \mathbb{E}_2^4 . The Gauss map of M_1^2 is defined by

$$\begin{aligned} G : M &\rightarrow \mathbb{H}_3^5(-1) \subset \mathbb{E}_4^6 \\ p &\mapsto G(p) = (x \wedge y)(p). \end{aligned}$$

We shall use the frame field $\{x, y, n_1, n_2\}$ defined in the previous subsection. This frame field generates the following frame of the Grassmanian manifold:

$$\{x \wedge y, x \wedge n_1, x \wedge n_2, y \wedge n_1, y \wedge n_2, n_1 \wedge n_2\},$$

for which we have

$$\begin{aligned} \langle x \wedge y, x \wedge y \rangle &= -1, & \langle x \wedge n_1, x \wedge n_1 \rangle &= 0, & \langle x \wedge n_2, x \wedge n_2 \rangle &= 0, \\ \langle y \wedge n_1, y \wedge n_1 \rangle &= 0, & \langle y \wedge n_2, y \wedge n_2 \rangle &= 0, & \langle n_1 \wedge n_2, n_1 \wedge n_2 \rangle &= -1, \\ \langle x \wedge n_1, y \wedge n_2 \rangle &= 1, & \langle x \wedge n_2, y \wedge n_1 \rangle &= 1, \end{aligned}$$

and all other scalar products are equal to zero.

Since $\langle x, x \rangle = \langle y, y \rangle = 0$, $\langle x, y \rangle = -1$, the Laplace operator $\Delta : C^\infty(M_1^2) \rightarrow C^\infty(M_1^2)$ of M_1^2 takes the form

$$\Delta\varphi = \tilde{\nabla}_x \tilde{\nabla}_y \varphi + \tilde{\nabla}_y \tilde{\nabla}_x \varphi - \tilde{\nabla}_{\nabla_x y} \varphi - \tilde{\nabla}_{\nabla_y x} \varphi$$

for any real valued function φ .

Hence, the Laplacian of the Gauss map is given by the formula

$$\Delta G = \tilde{\nabla}_x \tilde{\nabla}_y G + \tilde{\nabla}_y \tilde{\nabla}_x G - \tilde{\nabla}_{\nabla_x y} G - \tilde{\nabla}_{\nabla_y x} G.$$

By a direct computation, using (2), (4), (5), and (6), we obtain

$$(11) \quad \Delta G = -2Kx \wedge y + 2\kappa n_1 \wedge n_2 + 2\beta_2 x \wedge n_1 - 2\beta_1 y \wedge n_1.$$

The next proposition follows directly from formula (11).

Proposition 3.4. *Let M_1^2 be a quasi-minimal surface in the pseudo-Euclidean space \mathbb{E}_2^4 . Then, M_1^2 has harmonic Gauss map if and only if M_1^2 is a flat surface with parallel mean curvature vector field.*

Remark 3.5. See [27] for the classification of Lorentzian surfaces with parallel mean curvature vector in \mathbb{E}_2^4 .

Further we shall study quasi-minimal surfaces with pointwise 1-type Gauss map, i.e. the Laplacian of G satisfies (1) for a smooth non-vanishing function ϕ and a constant vector C .

Having in mind Lemma 3.1 and Lemma 3.3, from (11) we have the following proposition.

Proposition 3.6. *Let M_1^2 be a quasi-minimal surface in the pseudo-Euclidean space \mathbb{E}_2^4 . If M_1^2 is a non-flat surface with parallel mean curvature vector field, then it has proper pointwise 1-type Gauss map of first kind. In this case, (1) is satisfied for the smooth function $\phi = -2K$.*

Now, we shall give the complete classification of quasi-minimal surfaces with pointwise 1-type Gauss map.

Assume that M_1^2 has pointwise 1-type Gauss map. Then from (1) and (11) we get the equality

$$-2Kx \wedge y + 2\kappa n_1 \wedge n_2 + 2\beta_2 x \wedge n_1 - 2\beta_1 y \wedge n_1 = \phi(G + C), \quad \phi \neq 0,$$

which implies

$$(12a) \quad \langle C, x \wedge y \rangle = 1 + \frac{2K}{\phi},$$

$$(12b) \quad \langle C, n_1 \wedge n_2 \rangle = -\frac{2\kappa}{\phi},$$

$$(12c) \quad \langle C, x \wedge n_1 \rangle = 0,$$

$$(12d) \quad \langle C, y \wedge n_1 \rangle = 0,$$

$$(12e) \quad \langle C, x \wedge n_2 \rangle = -\frac{2\beta_1}{\phi},$$

$$(12f) \quad \langle C, y \wedge n_2 \rangle = \frac{2\beta_2}{\phi}.$$

3.3. Flat quasi-minimal surfaces with pointwise 1-type Gauss map. In this subsection we give the classification of flat quasi-minimal surfaces in \mathbb{E}_2^4 with pointwise 1-type Gauss map.

Let M_1^2 be a flat quasi-minimal surface with pointwise 1-type Gauss map, i.e., (1) is satisfied for a non-zero function ϕ and a constant vector C . Applying x and y to (12a) and using $K = 0$, we obtain $\beta_2 b = \beta_1 d = 0$.

Note that if M_1^2 has parallel mean curvature vector, then according to Proposition 3.4 the Gauss map is harmonic. Since we consider the non-harmonic case, we have $\beta_1^2 + \beta_2^2 \neq 0$. Hence, $b d = 0$, i.e. at least one of the functions b or d is zero. If we assume that $b = 0, d \neq 0$,

then we get $\varkappa = 0, \beta_1 = 0, \beta_2 = 0$. Applying y to (12e) we obtain $-1 = 0$, which is a contradiction. Similarly for the case $d = 0, b \neq 0$. Hence, the only possible case is $b = d = 0$. Thus, according to Remark 3.2 M_1^2 is congruent to the surface given by (7) for some function $\theta(u, v)$. Now, using the condition that the surface has pointwise 1-type Gauss map we will obtain conditions on $\theta(u, v)$.

We put $\eta_0 = (1, 0, 0, 1)$, $\eta_1 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ and $\eta_2 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

Theorem 3.7. *Let M_1^2 be a flat quasi-minimal surface in the pseudo-Euclidean space \mathbb{E}_2^4 . Then, M_1^2 has pointwise 1-type Gauss map if and only if it is congruent to the surface given by (7) for a smooth function θ satisfying*

$$(13) \quad \frac{\partial^2 \theta(u, v)}{\partial u \partial v} = (F \circ \psi)(u, v), \quad \psi(u, v) = \theta(u, v) + c_1 u + c_2 v,$$

where F is a non-constant function, c_1 and c_2 are constants. In this case, (1) is satisfied for the smooth function

$$(14) \quad \phi = (F' \circ \psi)$$

and the non-zero constant vector

$$(15) \quad C = c_1 \eta_0 \wedge \eta_2 - c_2 \eta_0 \wedge \eta_1 - \eta_1 \wedge \eta_2.$$

Proof. By a direct computation from (7) we get

$$\begin{aligned} x = z_u &= \theta_u \eta_0 + \eta_1, \\ y = z_v &= \theta_v \eta_0 + \eta_2, \end{aligned}$$

which imply

$$(16) \quad G = \eta_0 \wedge (\theta_u \eta_2 - \theta_v \eta_1) + \eta_1 \wedge \eta_2$$

and

$$(17) \quad \Delta G = \eta_0 \wedge (2\zeta_u \eta_2 - 2\zeta_v \eta_1),$$

where we put $\zeta = \frac{\partial^2 \theta}{\partial u \partial v}$.

From (1), (16) and (17) we obtain

$$\eta_0 \wedge \left(\left(2\frac{\zeta_u}{\phi} - \theta_u \right) \eta_2 - \left(2\frac{\zeta_v}{\phi} - \theta_v \right) \eta_1 \right) = C + \eta_1 \wedge \eta_2.$$

Since the right hand side of this equation is a constant vector, the vector field

$$\eta_0 \wedge \left(\left(2\frac{\zeta_u}{\phi} - \theta_u \right) \eta_2 - \left(2\frac{\zeta_v}{\phi} - \theta_v \right) \eta_1 \right)$$

is also constant. So, we obtain that the system of equations

$$\begin{aligned} 2\frac{\zeta_u}{\phi} - \theta_u &= c_1, \\ 2\frac{\zeta_v}{\phi} - \theta_v &= c_2 \end{aligned}$$

is satisfied for some constants c_1 and c_2 . The last two equations imply

$$\phi = 2\frac{\zeta_u}{c_1 + \theta_u} = 2\frac{\zeta_v}{c_2 + \theta_v}.$$

From this equation, one can see that the function ζ remains constant along the curve $\psi = c$, where ψ is the function given by the second equality in (13). Hence, we have proved that θ satisfies the first equality in (13).

Conversely, by a straightforward computation one can see that (1) is satisfied for the constant vector C and the smooth non-zero function F given by (14) and (15). \square

3.4. Quasi-minimal surfaces with flat normal connection. In this subsection, we focus on quasi-minimal surfaces with $\varkappa = 0$.

Let M_1^2 be a quasi-minimal surface with flat normal connection and pointwise 1-type Gauss map. Then, (12b) implies $\langle C, n_1 \wedge n_2 \rangle = 0$. Applying x and y to the last equation and using (12e) and (12f) we obtain

$$b\beta_2 = d\beta_1 = 0.$$

On the other hand, from (5) and (6) we have $K = 2bc$. If the Gauss curvature does not vanish, then $\beta_1 = \beta_2 = 0$ and hence M_1^2 has parallel mean curvature vector field. Combining this with Proposition 3.6, we get the following result.

Theorem 3.8. *Let M_1^2 be a quasi-minimal surface in the pseudo-Euclidean space \mathbb{E}_2^4 with flat normal connection and non-vanishing Gauss curvature. Then, M_1^2 has pointwise 1-type Gauss map if and only if it has parallel mean curvature vector field. In this case, M_1^2 has proper pointwise 1-type Gauss map of first kind.*

Remark 3.9. We would like to note that a classification of Lorentz surfaces with parallel mean curvature vector field in the pseudo-Euclidean space \mathbb{E}_2^4 is given in [27, Theorem 3.1]. Considering this theorem and its proof, one can see that there exist two families of quasi-minimal surfaces with proper pointwise 1-type Gauss map of first kind:

- (i) A non-flat CMC-surface lying in $\mathbb{S}_2^3(r^2)$ for some $r > 0$ such that the mean curvature vector H' of M in $\mathbb{S}_2^3(r^2)$ satisfies $\langle H', H' \rangle = -r^2$;
- (ii) A non-flat CMC-surface lying in $\mathbb{H}_1^3(-r^2)$ for some $r > 0$ such that the mean curvature vector H' of M in $\mathbb{H}_1^3(-r^2)$ satisfies $\langle H', H' \rangle = r^2$.

Conversely, any quasi-minimal surface with proper pointwise 1-type Gauss map of first kind belongs to one of the above two families.

3.5. Quasi-minimal surfaces with non-flat normal connection. In this subsection we focus on quasi-minimal surfaces with non-flat normal connection and pointwise 1-type Gauss map. Before we proceed, we would like to note that recent results show that there are no such surfaces if the ambient space is \mathbb{E}_1^4 or $\mathbb{S}_1^4(1)$ (see [31, 34, 35]).

First we prove the following proposition.

Proposition 3.10. *Let M_1^2 be a quasi-minimal surface in \mathbb{E}_2^4 with non-vanishing normal curvature. Then, M_1^2 has pointwise 1-type Gauss map if and only if there exists a local coordinate system (s, t) such that $\bar{x} = \partial_s$, $\bar{y} = \tilde{f}\partial_s + \partial_t$, n_1 and n_2 form a pseudo-orthonormal frame field of the tangent bundle of M_1^2 and the Levi-Civita connection satisfies*

$$\begin{aligned} (18a) \quad & \tilde{\nabla}_{\bar{x}}\bar{x} = \tilde{a}n_1, & \tilde{\nabla}_{\bar{y}}\bar{x} &= -\tilde{f}_s\bar{x} + n_1, \\ (18b) \quad & \tilde{\nabla}_{\bar{x}}\bar{y} = n_1, & \tilde{\nabla}_{\bar{y}}\bar{y} &= \tilde{f}_s\bar{y} + \tilde{c}n_1 + \tilde{d}n_2, \\ (18c) \quad & \tilde{\nabla}_{\bar{x}}n_1 = 0, & \tilde{\nabla}_{\bar{y}}n_1 &= -\tilde{d}\bar{x} + \tilde{\beta}_2n_1, \\ (18d) \quad & \tilde{\nabla}_{\bar{x}}n_2 = -\bar{x} - \tilde{a}\bar{y}, & \tilde{\nabla}_{\bar{y}}n_2 &= -\tilde{c}\bar{x} - \bar{y} - \tilde{\beta}_2n_2 \end{aligned}$$

where \tilde{a} , \tilde{c} , \tilde{d} , $\tilde{\beta}_2$ and \tilde{f} are smooth functions given by

$$(19a) \quad \tilde{a} = \lambda_3 \left(-\frac{2}{3}\lambda_1 s - \frac{2}{3}\lambda_2 \right)^{-3/2}, \quad \tilde{d} = \frac{1}{\lambda_3},$$

$$(19b) \quad \tilde{c} = -\frac{9}{\lambda_1^2} \left(-\frac{2}{3}\lambda_1 s - \frac{2}{3}\lambda_2 \right)^{1/2},$$

$$(19c) \quad \tilde{\beta}_2 = \frac{3}{\lambda_1} \left(-\frac{2}{3}\lambda_1 s - \frac{2}{3}\lambda_2 \right)^{-1/2},$$

$$(19d) \quad \tilde{f} = -\frac{9}{\lambda_1^2} \left(-\frac{2}{3}\lambda_1 s - \frac{2}{3}\lambda_2 \right)^{1/2} - \left(2\frac{\lambda'_3}{\lambda_3} - 3\frac{\lambda'_1}{\lambda_1} \right) s - \frac{\lambda'_2}{\lambda_1} - 2\frac{\lambda_2 \lambda'_3}{\lambda_1 \lambda_3} + 4\frac{\lambda_2 \lambda'_1}{\lambda_1 \lambda_1}$$

for some non-vanishing smooth functions λ_1, λ_3 and a smooth function λ_2 .

In this case, M_1^2 has Gauss curvature

$$(20) \quad K = \left(-\frac{2}{3}\lambda_1 s - \frac{2}{3}\lambda_2 \right)^{-3/2}.$$

Moreover, (1) is satisfied for

$$(21) \quad \phi = -4K \quad \text{and} \quad C = -\frac{1}{2} \left(\bar{x} \wedge \bar{y} + n_1 \wedge n_2 + \frac{\tilde{\beta}_2}{K} \bar{x} \wedge n_1 \right).$$

Proof. Assume that the normal curvature \varkappa is non-vanishing and M_1^2 has pointwise 1-type Gauss map. Then, (1) is satisfied for a constant vector C and a smooth non-vanishing function ϕ such that formulas (12) hold true. Applying x to (12c) and y to (12d), we obtain

$$(22a) \quad b(\langle C, x \wedge y \rangle + \langle C, n_1 \wedge n_2 \rangle) = 0,$$

$$(22b) \quad d(\langle C, x \wedge y \rangle - \langle C, n_1 \wedge n_2 \rangle) = 0.$$

Note that if $b^2 + d^2 = 0$ at a point p , then (6) implies $\varkappa(p) = 0$ which contradicts the assumption that \varkappa is non-vanishing. If both b and d are non-zero, then $\langle C, x \wedge y \rangle = \langle C, n_1 \wedge n_2 \rangle = 0$ and from (12b) we get $\varkappa = 0$, a contradiction. Hence, $b = 0, d \neq 0$ or $b \neq 0, d = 0$. Therefore, replacing x and y if necessary, we may assume that $d \neq 0$ and $b = 0$. Thus, (22b) gives $\langle C, x \wedge y \rangle = \langle C, n_1 \wedge n_2 \rangle$ and equations (5), (6) imply $K = \varkappa = ad$. Therefore, (12a) and (12b) imply

$$(23) \quad \langle C, x \wedge y \rangle = \langle C, n_1 \wedge n_2 \rangle = \frac{1}{2} \quad \text{and} \quad \phi = -4K.$$

On the other hand, from $y(\langle C, x \wedge y \rangle) = 0$ we obtain $d\langle C, x \wedge n_2 \rangle = 0$ which gives $\beta_1 = 0$ because of (12e). Therefore, combining (12) and (23), we obtain

$$(24) \quad C = -\frac{1}{2} \left(x \wedge y + n_1 \wedge n_2 + \frac{\beta_2}{K} x \wedge n_1 \right).$$

Next, we define a local coordinate system (s, t) in the following way:

$$s = s(u, v) = \int_{u_0}^u f^2(\tau, v) d\tau, \quad t = v.$$

Note that we have

$$\partial_u = f^2 \partial_s \quad \text{and} \quad \partial_v = \tilde{f} \partial_s + \partial_t,$$

where $\tilde{f} = \frac{\partial}{\partial v} \left(\int_{u_0}^u f^2(\tau, v) d\tau \right)$, which give

$$\langle \partial_s, \partial_s \rangle = 0, \quad \langle \partial_s, \partial_t \rangle = -1, \quad \langle \partial_t, \partial_t \rangle = 2\tilde{f}.$$

Thus, the metric tensor of M_1^2 with respect to the new coordinate system (s, t) takes the form

$$g = -(ds \otimes dt + dt \otimes ds) + 2\tilde{f}dt \otimes dt.$$

It is easy to calculate that

$$(25a) \quad \nabla_{\partial_s} \partial_s = 0,$$

$$(25b) \quad \nabla_{\partial_s} \partial_t = \nabla_{\partial_t} \partial_s = -\tilde{f}_s \partial_s,$$

$$(25c) \quad \nabla_{\partial_t} \partial_t = \tilde{f}_s \partial_t + (2\tilde{f}\tilde{f}_s - \tilde{f}_t) \partial_s.$$

Moreover, $\bar{x} = \partial_s = \frac{1}{f}x$ and $\bar{y} = \tilde{f}\partial_s + \partial_t = fy$. Hence, $\{\bar{x}, \bar{y}\}$ form a pseudo-orthonormal frame field of the tangent bundle of M_1^2 . Using (25) and taking into account that $b = \beta_1 = 0$, we get that the Levi-Civita connection of M_1^2 satisfies (18) for the functions $\tilde{a} = a/f^2$, $\tilde{c} = f^2c$, $\tilde{d} = f^2d$ and $\tilde{\beta}_2 = f\beta_2$. Now, equality (24) gives the second equality in (21).

With respect to the new coordinate system (s, t) the integrability conditions (4) become

$$(26a) \quad \tilde{c}_s = \tilde{\beta}_2,$$

$$(26b) \quad \tilde{d}_s = 0,$$

$$(26c) \quad \tilde{f}\tilde{a}_s + \tilde{a}_t = -\tilde{a}\tilde{\beta}_2 - 2\tilde{a}\tilde{f}_s,$$

$$(26d) \quad (\tilde{\beta}_2)_s = \tilde{a}\tilde{d} = K = \varkappa,$$

$$(26e) \quad \tilde{f}_{ss} = \tilde{a}\tilde{d} = K = \varkappa.$$

Since $d \neq 0$, from (26b) we get the second equality in (19a) for a non-vanishing smooth function $\lambda_3 = \lambda_3(t)$. Furthermore, combining (26a), (26d), and (26e), we get

$$(27) \quad \tilde{c} = \tilde{f} + \lambda_4 s + \lambda_5$$

and

$$(28) \quad \tilde{\beta}_2 - \tilde{f}_s = \lambda_4$$

for some smooth functions $\lambda_4 = \lambda_4(t)$ and $\lambda_5 = \lambda_5(t)$.

Next, using (25) we obtain

$$\begin{aligned} \tilde{\nabla}_{\bar{x}} C &= -\frac{1}{2} \left(2 + \bar{x} \left(\frac{\tilde{\beta}_2}{K} \right) \right) \bar{x} \wedge n_1, \\ \tilde{\nabla}_{\bar{y}} C &= -\frac{1}{2} \left(2\tilde{c} + \bar{y} \left(\frac{\tilde{\beta}_2}{K} \right) + \frac{\tilde{\beta}_2}{K} (\tilde{\beta}_2 - \tilde{f}_s) \right) \bar{x} \wedge n_1. \end{aligned}$$

Since C is a constant vector, we have $\tilde{\nabla}_{\bar{x}} C = 0$ and $\tilde{\nabla}_{\bar{y}} C = 0$. So, we get the following two equations:

$$(29a) \quad \bar{x} \left(\frac{\tilde{\beta}_2}{K} \right) = -2,$$

$$(29b) \quad \bar{y} \left(\frac{\tilde{\beta}_2}{K} \right) + \frac{\tilde{\beta}_2}{K} (\tilde{\beta}_2 - \tilde{f}_s) + 2\tilde{c} = 0.$$

From (29a) using (26d) we get $\tilde{\beta}_2 K_s = 3K^2$. Differentiating the last equality with respect to s , we obtain the equation

$$3KK_{ss} = 5K_s^2,$$

whose solution is given by

$$K = \left(-\frac{2}{3}\lambda_1(t)s - \frac{2}{3}\lambda_2(t) \right)^{-3/2}$$

for some smooth functions $\lambda_1 = \lambda_1(t) \neq 0$ and $\lambda_2 = \lambda_2(t)$. The function $\tilde{\beta}_2$ is expressed as $\tilde{\beta}_2 = \frac{3}{\lambda_1(t)}K^{1/3}$, which implies (19c). Combining (20) and (26e) with the second equality in (19a), we obtain that \tilde{a} is expressed as given by the first equality in (19a).

Furthermore, from (19c) and (28) we get

$$(30) \quad \tilde{f} = -\frac{9}{\lambda_1^2} \left(-\frac{2}{3}\lambda_1(t)s - \frac{2}{3}\lambda_2(t) \right)^{1/2} - \lambda_4(t)s + \lambda_6(t)$$

for a smooth function $\lambda_6 = \lambda_6(t)$.

Next, using (29b) and taking into consideration (19c), (27), (28) and (20), we get that the function $\lambda_5(t)$ is expressed as follows:

$$(31) \quad \lambda_5 = \left(\frac{\lambda_2}{\lambda_1} \right)' + \frac{\lambda_2 \lambda_4}{\lambda_1}.$$

Further, using (19a), (19c), (30), and (26c), we obtain

$$s \left(\frac{1}{3}\lambda_1 \lambda_3 \lambda_4 + \lambda_1' \lambda_3 - \frac{2}{3}\lambda_1 \lambda_3' \right) + \left(\lambda_1 \lambda_3 \lambda_6 + \lambda_2' \lambda_3 - \frac{2}{3}\lambda_2 \lambda_3' + \frac{4}{3}\lambda_2 \lambda_3 \lambda_4 \right) = 0,$$

which implies the following expressions for the functions $\lambda_4(t)$ and $\lambda_6(t)$:

$$(32) \quad \lambda_4 = 2\frac{\lambda_3'}{\lambda_3} - 3\frac{\lambda_1'}{\lambda_1},$$

$$(33) \quad \lambda_6 = -\frac{\lambda_2'}{\lambda_1} - \frac{2\lambda_2 \lambda_3'}{\lambda_1 \lambda_3} + \frac{4\lambda_1' \lambda_2}{\lambda_1^2}.$$

Now, from (27), (30), (31), and (32) we obtain (19b). Finally, taking into consideration (30), (32), and (33) we obtain that the function \tilde{f} is expressed as given in (19d). Hence, the proof of the necessary condition is completed.

In order to prove the sufficient condition we assume that there exists a coordinate system (s, t) such that (18) and (19) are satisfied. By a straightforward computation using (11) one can check that G satisfies (1) for the non-constant function $\phi = -4 \left(-\frac{2}{3}\lambda_1 s - \frac{2}{3}\lambda_2 \right)^{-3/2}$

and vector $C = -\frac{1}{2} \left(\bar{x} \wedge \bar{y} + n_1 \wedge n_2 + \frac{\tilde{\beta}_2}{K} \bar{x} \wedge n_1 \right)$. A further calculation yields that C is

a non-zero constant vector. Hence, M_1^2 has proper pointwise 1-type Gauss map of second kind. \square

Corollary 3.11. *Let M_1^2 be a quasi-minimal surface with non-flat normal connection in the pseudo-Euclidean space \mathbb{E}_2^4 . If the Gauss map of M_1^2 satisfies (1), then M_1^2 has proper pointwise 1-type Gauss map of second kind.*

In the following theorem, we obtain a parametrization for the surface described in Proposition 3.10. This completes the classification of quasi-minimal surfaces in \mathbb{E}_2^4 with non-flat normal connection and pointwise 1-type Gauss map.

Theorem 3.12. *Let M_1^2 be a quasi-minimal surface in \mathbb{E}_2^4 with non-vanishing normal curvature. Then, M_1^2 has pointwise 1-type Gauss map if and only if it is congruent to the surface given by*

$$(34) \quad z(s, t) = -s\lambda_3(t)n_1'(t) - \frac{3\sqrt{6}\lambda_3(t)\sqrt{-s\lambda_1(t)}}{\lambda_1^2(t)}n_1(t) + \xi(t).$$

for some smooth functions $\lambda_1 = \lambda_1(t)$, $\lambda_3 = \lambda_3(t)$ and some \mathbb{E}_2^4 -valued smooth functions $n_1(t)$, $\xi(t)$ satisfying the equations

$$(35) \quad \langle n_1, n_1 \rangle = \langle n_1', n_1' \rangle = \langle n_1, \xi' \rangle = \langle \xi', \xi' \rangle = 0, \quad \langle n_1', \xi' \rangle = \frac{1}{\lambda_3},$$

$$(36) \quad n_1'' - \left(\frac{\lambda_3'}{\lambda_3} - \frac{3\lambda_1'}{\lambda_1} \right) n_1' + \frac{1}{\lambda_3} n_1 = 0,$$

and

$$(37) \quad \xi''' + \left(\frac{3\lambda_3'}{\lambda_3} - \frac{3\lambda_1'}{\lambda_1} \right) \xi'' + \frac{-3\lambda_1(\lambda_1'\lambda_3' + \lambda_3\lambda_1'') + 3\lambda_3\lambda_1'^2 + \lambda_1^2(2\lambda_3'' + 1)}{\lambda_1^2\lambda_3} \xi' = \zeta,$$

where $\zeta = \zeta(t)$ is the \mathbb{E}_2^4 -valued function given by

$$(38) \quad \zeta = 81 \frac{8\lambda_1'^2\lambda_3^2 - 2\lambda_1\lambda_1''\lambda_3^2 + \lambda_1^2\lambda_3\lambda_3'' - 7\lambda_1\lambda_1'\lambda_3\lambda_3' + \lambda_1^2\lambda_3'^2}{\lambda_1^5\lambda_3} n_1 + 162 \frac{\lambda_1\lambda_3' - 2\lambda_3\lambda_1'}{\lambda_1^4} n_1'.$$

Proof. Let M_1^2 be the surface described in Proposition 3.10 for some smooth functions $\lambda_1(t)$ and $\lambda_3(t)$. Then, the first equation in (18c) implies $n_1 = n_1(t)$. Thus, the first equation in (18a) and (19a) give the differential equation

$$z_{ss} = \lambda_3 \left(-\frac{2}{3}\lambda_1 s - \frac{2}{3}\lambda_2 \right)^{-3/2} n_1(t)$$

which implies

$$z_s = \frac{3\lambda_3}{\lambda_1} \left(-\frac{2}{3}\lambda_1 s - \frac{2}{3}\lambda_2 \right)^{-1/2} n_1(t) + \xi_1(t)$$

for some \mathbb{E}_2^4 -valued smooth function $\xi_1(t)$. Hence, the position vector $z(s, t)$ is given by

$$(39) \quad z(s, t) = -\frac{3\sqrt{6}\lambda_3\sqrt{-s\lambda_1 - \lambda_2}}{\lambda_1^2} n_1(t) + s\xi_1(t) + \xi(t)$$

where $\xi(t)$ is a \mathbb{E}_2^4 -valued smooth function. Without loss of generality, we may assume that $\lambda_2 = 0$ by re-defining s .

Now, using (18c), (19a) and (19c) we obtain $\xi_1(t) = -\lambda_3 n_1'(t)$. Combining the last equality with (39) we get that M_1^2 is congruent to the surface parametrized by (34).

Now, from the first equation in (18b) we have

$$\frac{\partial}{\partial s} (\tilde{f}z_s + z_t) = n_1.$$

The last equation together with (30) and (34) imply (36). Note that, since $\bar{x} = z_s$ and $\bar{y} = \tilde{f}z_s + z_t$ form a pseudo-orthonormal frame field of the tangent bundle of M_1^2 , (34) and (36) imply (35).

On the other hand, from the second equality in (18b) we have

$$n_2 = \frac{1}{\tilde{d}} \left(\tilde{\nabla}_{\bar{y}} \bar{y} - \tilde{f}_s \bar{y} - \tilde{c} n_1 \right).$$

Using (19), (36) and $\lambda_2 = 0$, we compute the right-hand side of this equality and obtain

$$\begin{aligned} n_2 = & \frac{1}{2s\lambda_1^2} \left(2s\lambda_1(2\lambda_3'\lambda_1 - 3\lambda_1'\lambda_3 + 3\sqrt{6}\sqrt{-s\lambda_1}\lambda_3) \xi' + \lambda_3\xi'' + \frac{\lambda_3}{\lambda_1^3} (s\lambda_1^3 - 27\lambda_3) n_1' \right. \\ & \left. + \frac{3\lambda_3}{2s\lambda_1^5} \left(54s\lambda_1(2\lambda_3\lambda_1' - \lambda_1\lambda_3') + \sqrt{6}\sqrt{-s\lambda_1}(s\lambda_1^3 - 27\lambda_3) \right) n_1 \right). \end{aligned}$$

The last equality together with the second equality in (18d) and (36) imply

$$\begin{aligned} \xi''' + 3 \left(\frac{\lambda_3'}{\lambda_3} - \frac{\lambda_1'}{\lambda_1} \right) \xi'' + \left(\frac{2\lambda_3''}{\lambda_3} - \frac{3\lambda_3'\lambda_1'}{\lambda_1\lambda_3} - \frac{3\lambda_1''}{\lambda_1} + \frac{3\lambda_1'^2}{\lambda_1^2} + \frac{1}{\lambda_3} \right) \xi' = \\ = 81 \left(\frac{8\lambda_1'^2\lambda_3}{\lambda_1^5} - \frac{2\lambda_3\lambda_1''}{\lambda_1^4} + \frac{\lambda_3''}{\lambda_1^3} - \frac{7\lambda_1'\lambda_3'}{\lambda_1^4} + \frac{\lambda_3'^2}{\lambda_1^3\lambda_3} \right) n_1 + 162 \left(\frac{\lambda_3'}{\lambda_1^3} - \frac{2\lambda_3\lambda_1'}{\lambda_1^4} \right) n_1'. \end{aligned}$$

Denoting by ζ the vector field given in (38) we obtain (37). Hence, the proof of the necessary condition is completed.

The converse follows by a direct computation. \square

Below we present an explicit example of a quasi-minimal surface with non-flat normal connection and pointwise 1-type Gauss map.

Example 3.13. Let \mathcal{M} be the surface given by

$$\begin{aligned} z(s, t) = & \left(-4s^{1/2} \cos t + s \sin t + \frac{1}{2} \cos t, -4s^{1/2} \sin t - s \cos t + \frac{1}{2} \sin t, \right. \\ & \left. -4s^{1/2} \sin t - s \cos t - \frac{1}{2} \sin t, -4s^{1/2} \cos t + s \sin t - \frac{1}{2} \cos t \right) \end{aligned}$$

in the pseudo-Euclidean space \mathbb{E}_2^4 . Calculating the tangent vector fields z_s and z_t we get $\langle z_s, z_s \rangle = 0$, $\langle z_s, z_t \rangle = -1$, $\langle z_t, z_t \rangle = -8s^{1/2}$. Hence, \mathcal{M} is a Lorentz surface in \mathbb{E}_2^4 .

We consider the following normal vector field $n_1 = (\cos t, \sin t, \sin t, \cos t)$. Note that $\langle n_1, n_1 \rangle = 0$. Hence, there exists a normal vector field n_2 such that $\langle n_2, n_2 \rangle = 0$ and $\langle n_1, n_2 \rangle = -1$. Now, we consider the following pseudo-orthonormal tangent frame field

$$\bar{x} = z_s, \quad \bar{y} = -4s^{1/2} z_s + z_t.$$

By a straightforward computation one can see that the mean curvature vector field is $H = -n_1$. Hence, \mathcal{M} is a quasi-minimal surface. Direct computations show that

$$\tilde{\nabla}_{\bar{x}} n_1 = 0, \quad \tilde{\nabla}_{\bar{y}} n_1 = -\bar{x} - 2s^{-1/2} n_1,$$

which imply $\tilde{\beta}_1 = 0$, $\tilde{\beta}_2 = -2s^{-1/2}$. So, \mathcal{M} is a surface with non-parallel mean curvature vector field. The Gauss curvature K and the normal curvature \varkappa are given by the expressions:

$$K = s^{-3/2}, \quad \varkappa = s^{-3/2}.$$

Hence, \mathcal{M} is a quasi-minimal surface with non-flat normal connection. By a straightforward computation we obtain that (1) is satisfied for the function $\phi = -4s^{-3/2}$ and the constant vector $C = -\frac{1}{2}(\bar{x} \wedge \bar{y} - 2s\bar{x} \wedge n_1 + n_1 \wedge n_2)$. So, \mathcal{M} is of proper pointwise 1-type Gauss map of second type.

Note that the Levi-Civita connection of \mathcal{M} satisfies (18) for the functions $\tilde{a} = s^{-3/2}$, $\tilde{d} = 1$, $\tilde{\beta}_2 = -2s^{-1/2}$, $\tilde{c} = \tilde{f} = -4s^{1/2}$, which can be obtained by putting $\lambda_1 = -\frac{3}{2}$, $\lambda_2 = 0$, $\lambda_3 = 1$ in (19).

Remark 3.14. We would like to note that in the Minkowski space \mathbb{E}_1^4 all marginally trapped surfaces with pointwise 1-type Gauss map have flat normal connection, while in the pseudo-Euclidean space \mathbb{E}_2^4 there exist quasi-minimal surfaces with non-flat normal connection and pointwise 1-type Gauss map.

Remark 3.15. As far as we know, all examples of quasi-minimal surfaces in \mathbb{E}_2^4 known till now in the literature are surfaces with parallel mean curvature vector field. Here we give an explicit example of a quasi-minimal surface with non-parallel mean curvature vector field.

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